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Properties of harmonic conjugates

ABSTRACT. We give a new proof of Hardy and Littlewood theorem concerning harmonic conjugates of functions $u$ such that $\int_D |u(z)|^p dA(z) < \infty$, $0 < p \leq 1$. We also obtain an inequality for integral means of such harmonic functions $u$.

Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ and $dA$ be the Lebesgue measure normalized so that $A(D) = 1$. The harmonic Hardy space $h^p$, $0 < p < \infty$, consists of all real-valued functions $u$ harmonic in $D$ whose integral means

$$M_p(r, u) = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |u(re^{i\theta})|^p d\theta \right\}^{\frac{1}{p}}$$

are bounded. The harmonic Bergman space $a^p$ is the collection of all real-valued harmonic functions $u$ in $D$ for which the integral

$$||u||_p^p = \int_D |u(z)|^p dA(z)$$

is finite. For a real-valued function $u$ harmonic in $D$ we define the harmonic conjugate as the function $\nu$ with $\nu(0) = 0$ such that $f = u + i\nu$ is analytic in $D$. By the theorem of M. Riesz, if $1 < p < \infty$ and $u \in h^p$, then $\nu \in h^p$ and $M_p(r, \nu) \leq CM_p(r, u)$ where $C$ depends only on $p$. For $0 < p \leq 1$ or $p = \infty$ the theorem fails.

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It follows immediately from the theorem of M. Riesz that for every $p$ in the range $1 < p < \infty$ if $u \in a^p$, then $v \in a^p$ and $\|v\|_p \leq C\|u\|_p$. However, in the space $a^p$ the last inequality holds also for $0 < p \leq 1$. This result was first stated by Hardy and Littlewood [4] and its proof was indicated there. Thus the following theorem holds.

**Theorem HL.** Let $0 < p < \infty$. If $u \in a^p$, then its conjugate $v \in a^p$ and $\|v\|_p \leq C\|u\|_p$, where $C$ depends only on $p$.

In [4] Watanabe presented the proof of the above theorem, when $0 < p \leq 1$. There are some gaps and the proof seems to be incomplete. For example the inequality in line 9 from the above on page 53 is not proved. We note that in the case when $0 < p < 1$ and $u$ is harmonic in $D$ the integral mean $\overline{M}_p(r, u)$ need not be monotonically increasing function of $r$. Moreover, the application of Lemma 4 in [1] at the end of the proof is not explained. In this paper we give a complete detailed proof of Theorem HL for the case $0 < p \leq 1$, shorter than that in [4]. Throughout this paper $C$ denotes a general positive constant which may differ from line to line.

**Proof of Theorem HL for the case when $0 < p \leq 1$.** Let $f = u + iv$ be analytic in $D$ and assume that $v(0) = 0$. We start with the following inequality proved in [1] p. 411.

\begin{equation}
\sigma |zf'(z)| \leq \eta^{-1} (|u(r+h, \theta)| + |u(r, \theta + h)| + 2|u(r, \theta)|) + A\mu \sigma \eta,
\end{equation}

where $z = re^{i\theta}$, $0 < r < 1$, $u(r, \theta) = u(re^{i\theta})$, $\sigma = \sigma(r) = \sqrt{r} - r$, $h = \eta \sigma$, $A = \sum_{m=2}^{\infty} 2^m \eta^{m-2} = 4/(1 - 2\eta)$, $\eta$ is any positive number less than $\frac{1}{4}$. Moreover, $\mu = \mu(r, \theta) = \max_{\gamma} |f'(z)|$ and $\gamma$ denotes the circle centered at the point $re^{i\theta}$ and the radius $\sigma$.

Since $0 < p \leq 1$, we get from (1)

\begin{align}
\sigma(r)^p & \frac{1}{2\pi} \int_{0}^{2\pi} r^p |f'(re^{i\theta})|^{p} d\theta \\
& \leq \eta^{-p} \left( \frac{1}{2\pi} \int_{0}^{2\pi} |u(r+h, \theta)|^{p} d\theta \right) \\
& + \frac{1}{2\pi} \int_{0}^{2\pi} |u(r, \theta + h)|^{p} d\theta + 2\mu \frac{1}{2\pi} \int_{0}^{2\pi} |u(r, \theta)|^{p} d\theta \\
& + (A\sigma \eta)^p \frac{1}{2\pi} \int_{0}^{2\pi} (r\mu)^{p} d\theta.
\end{align}

It was shown in [1] p. 411 that

\begin{equation}
\frac{1}{2\pi} \int_{0}^{2\pi} (r\mu)^{p} d\theta \leq C \frac{1}{2\pi} \int_{0}^{2\pi} r^2 |f'(re^{i\theta})|^{p} d\theta.
\end{equation}
Moreover, an easy calculation shows that \( \sigma(r) \leq 4\sigma(r^{\frac{1}{4}}) \). Now multiplying both sides of inequality (2) by \( 2r \) and integrating with respect \( r \) give

\[
\frac{1}{\pi} \int_0^1 \int_0^{2\pi} \sigma(r)r^p |f'(re^{i\theta})|^p d\theta dr \\
\leq \eta^{-p} \left( \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |u(r+h, \theta)|^p d\theta dr \right) + (2^p + 1) \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |u(r, \theta)|^p d\theta dr \\
+ C\eta^p \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \sigma(r^{\frac{1}{4}}) r^{\frac{p}{4}} |f'(r^{\frac{1}{4}} e^{i\theta})|^p d\theta dr.
\]

Substituting \( t^4 = r \) in the last integral yields

\[
\frac{1}{\pi} \int_0^1 \int_0^{2\pi} \sigma(r)r^p |f'(re^{i\theta})|^p d\theta dr \\
\leq \eta^{-p} \left( \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |u(r+h, \theta)|^p d\theta dr \right) + (2^p + 1) \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |u(r, \theta)|^p d\theta dr \\
+ C\eta^p \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \sigma(t^4) t^p |f'(te^{i\theta})|^p d\theta dt.
\]

(3)

It is clear that \( r + h = r + \eta(\sqrt{r} - r) < 1 \) on \( 0 < r < 1 \) and \( 0 < \eta < \frac{1}{4} \). Moreover, the function \( g(r) = r + \eta(\sqrt{r} - r) \) is increasing in the interval \( 0 < r < 1 \). Substituting \( r + h = t^2 \) in the first integral on the right hand side of (3) we get

\[
\frac{1}{\pi} \int_0^1 \int_0^{2\pi} |u(r+h, \theta)|^p d\theta dr \\
= \frac{2}{2(1-\eta)} \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |u(t^2, \theta)|^p \left( \frac{-\eta + \sqrt{\eta^2 + 4(1-\eta)t^2}}{2(1-\eta)} \right)^2 \\
\times \left( \frac{-\eta}{\sqrt{\eta^2 + 4(1-\eta)t^2}} + 1 \right) td\theta dt \\
\leq \frac{4}{2(1-\eta)} \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |u(t^2, \theta)|^p \left( \frac{-\eta + \sqrt{\eta^2 + 4(1-\eta)t^2}}{2(1-\eta)} \right)^2 \\
\times \left( \frac{-\eta}{2-\eta} + 1 \right) d\theta dt.
\]
\[ \frac{4}{2 - \eta} \int_0^1 \int_0^{2\pi} |u(t^2, \theta)|^p \left( -\eta + \frac{\sqrt{\eta^2 + 4(1 - \eta)t^2}}{2(1 - \eta)} \right)^2 d\theta dt \]

\[ \leq \frac{4}{2 - \eta} \int_0^1 \int_0^{2\pi} |u(t^2, \theta)|^p \left( -\eta + \eta + \sqrt{4(1 - \eta)t^2} \right)^2 d\theta dt \]

\[ = \frac{4}{(2 - \eta)(1 - \eta)} \int_0^1 \int_0^{2\pi} |u(t^2, \theta)|^p t^2 d\theta dt \]

\[ = \frac{2}{(2 - \eta)(1 - \eta)} \int_0^1 \int_0^{2\pi} |u(t, \theta)|^p d\theta dt. \]

By the assumption \( u \in a^p \) and (3) we get

\[ \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \sigma(r)r^p |f'(re^{i\theta})|^p d\theta dr \]

\[ \leq \frac{1}{\eta^p} \left( \frac{2}{(2 - \eta)(1 - \eta)} + 2^p + 1 \right) ||u||_{a^p}^p \]

\[ + C\eta^p \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \sigma(t)t^p |f'(te^{i\theta})|^p d\theta dt. \]

Now choosing \( \eta \) so that \( \eta < C^{-\frac{1}{p}} \) we get

\[ (1 - C\eta^p) \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \sigma(r)r^p |f'(re^{i\theta})|^p d\theta dr \leq C||u||_{a^p}^p. \]

We note that the convergence of the above integral implies the convergence of

\[ \frac{1}{\pi} \int_0^1 \int_0^{2\pi} (1 - r)^p |f'(re^{i\theta})|^p d\theta dr, \]

which means that \( f \in A^p \), see e.g. Lemma 4 in [4]. \( \square \)

**Corollary.** If \( u \in a^p, \ u(0) = 0, \ 0 < p \leq 1, \) then

\[ M_p(r, u) \leq C \frac{||u||_{a^p}}{(1 - r)^\frac{1}{p}}, \]

where a constant \( C \) depends only on \( p \).

**Proof.** Let \( f \) and \( \sigma \) be as in our proof of Theorem HL and assume that \( f(0) = 0. \) It is clear that the function \( \sigma \) is monotonically increasing in \( (0, \frac{1}{4}) \) and monotonically decreasing in \( (\frac{1}{4}, 1) \). Since \( M_p(r, f') \) is increasing
function of $r$ on $(0, 1)$, using the Chebyshev inequality (see e.g. [3]) we get
\[
\int_0^1 \int_0^{2\pi} \sigma(r)^p r^p |f'(re^{i\theta})|^p r d\theta dr \\
= \int_0^{\frac{1}{4}} \int_0^{2\pi} \sigma(r)^p r^p |f'(re^{i\theta})|^p r d\theta dr + \int_{\frac{1}{4}}^1 \int_0^{2\pi} \sigma(r)^p r^p |f'(re^{i\theta})|^p r d\theta dr \\
\geq C \int_0^{\frac{1}{4}} \int_0^{2\pi} |f'(re^{i\theta})|^p r d\theta dr + \frac{1}{8^p} \int_{\frac{1}{4}}^1 \int_0^{2\pi} (1 - \sqrt{r})^p |f'(re^{i\theta})|^p r d\theta dr \\
\geq C \int_0^{\frac{1}{4}} \int_0^{2\pi} (1 - r)^p |f'(re^{i\theta})|^p r d\theta dr \geq C \int_0^{1} \int_0^{2\pi} |f(re^{i\theta})|^p r d\theta dr,
\]
where the last inequality follows from e.g. Lemma 4 in [4]. Thus
\[
M_p(r, u)(1 - r) \leq M_p(r, f)(1 - r) \leq \int_r^1 \frac{1}{2\pi} \int_0^{2\pi} |f(te^{i\theta})|^p d\theta dt \leq C ||u||_a^p.
\]

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References


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