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The natural operators lifting vector fields
to the bundle of affinors

Abstract. All natural operators $T \mapsto T(T \otimes T^*)$ lifting vector fields $X$ from $n$-dimensional manifolds $M$ to vector fields $B(X)$ on the bundle of affinors $TM \otimes T^*M$ are described.

1. Introduction. In [3], the second author studied the problem how a 1-form $\omega$ on an $n$-manifold $M$ induces a 1-form $B(\omega)$ on $TM \otimes T^*M$. This problem was reflected in natural operators $B : T^* \mapsto T^*(T \otimes T^*)$ over $n$-manifolds. It is proved that the set of natural operators $T^* \mapsto T^*(T \otimes T^*)$ over $n$-manifolds is a free $C^\infty(\mathbb{R}^n)$-module of dimension $2n$, and there is presented a basis of this module.

In this note we study a similar problem how a vector field $X$ on an $n$-manifold $M$ induces a vector field $B(X)$ on $TM \otimes T^*M$. This problem is reflected in natural operators $T \mapsto T(T \otimes T^*)$ over $n$-manifolds. We prove that the set of natural operators $T \mapsto T(T \otimes T^*)$ over $n$-manifolds is a free $C^\infty(\mathbb{R}^n)$-module of dimension $n + 1$. We construct a basis of this module.

We recall that a natural operator $B : T \mapsto T(T \otimes T^*)$ over $n$-manifolds is an $\mathcal{M}_{f_n}$-invariant family of regular operators

$$B : \mathcal{X}(M) \rightarrow \mathcal{X}(TM \otimes T^*M)$$

2000 Mathematics Subject Classification. 58A20, 53A55.

Key words and phrases. Natural bundles, natural operators.
for all $n$-manifolds $M$. The invariance means that if vector fields $X_1$ on $M$ and $X_2$ on $N$ are $\varphi$-related for some local diffeomorphism $\varphi : M \to N$ between $n$-manifolds then the vector fields $B(X_1)$ and $B(X_2)$ are $T\varphi \otimes T^*\varphi$-related. The regularity means that $B$ transforms smoothly parametrized families of vector fields into smoothly parametrized families of vector fields.

From now on $x^1, \ldots, x^n$ are the usual coordinates on $\mathbb{R}^n$ and $\partial_i = \frac{\partial}{\partial x^i}$ for $i = 1, \ldots, n$ are the canonical vector fields on $\mathbb{R}^n$.

All manifolds and maps are assumed to be of class $C^\infty$.

2. Examples of natural operators $T \leadsto T(T \otimes T^*)$.

Example 2.1. Let $X$ be a vector field on an $n$-manifold $M$. Let $T \otimes T^*X$ be the flow lifting of $X$ to $TM \otimes T^*M$. More precisely, if $\varphi_t$ is the flow of $X$, then $T\varphi_t \otimes T^*\varphi_t$ is the flow of $T \otimes T^*X$. The correspondence $T \otimes T^* : T \leadsto T(T \otimes T^*)$ given by $X \to T \otimes T^*X$ is a natural operator (called the flow operator) in question.

Example 2.2. For $k = 0, \ldots, n-1$ we have the canonical vector field $L^k$ on $TM \otimes T^*M$ such that

$$L^k(A) = \left. \frac{d}{dt} \right|_0 (A + tA^k), \quad A \in \text{End}(T_xM) = T_xM \otimes T_x^*M, \quad x \in M,$$

where $A^k$ is the $k$-th power of $A$ ($A^0 = id$). The vector field $L^k$ will be called the $k$-th Liouville vector field on $TM \otimes T^*M$ ($L^1$ is the classical Liouville vector field on $TM \otimes T^*M$). The correspondence $L^k : T \leadsto T(T \otimes T^*)$ is a natural operator in question.

3. The $C^\infty(\mathbb{R}^n)$-module of natural operators $T \leadsto T(T \otimes T^*)$ over $n$-manifolds. If $L : V \to V$ is an endomorphism of an $n$-dimensional vector space $V$ then $a_1(L), \ldots, a_n(L)$ denote the coefficient of the characteristic polynomial

$$W_L(\lambda) = \det(\lambda id_V - L) = \lambda^n + a_1(L)\lambda^{n-1} + \cdots + a_{n-1}(L)\lambda + a_n(L).$$

Thus for every $n$-manifold $M$ we have maps $a_1, \ldots, a_n : TM \otimes T^*M \to \mathbb{R}$ (as $T_xM \otimes T_x^*M = \text{End}(T_xM)$).

The vector space of all natural operators $B : T \leadsto T(T \otimes T^*)$ over $n$-manifolds is additionally a module over the algebra $C^\infty(\mathbb{R}^n)$ of smooth maps $\mathbb{R}^n \to \mathbb{R}$. Actually given a smooth map $f : \mathbb{R}^n \to \mathbb{R}$ and a natural operator $B : T \leadsto T(T \otimes T^*)$ we have natural operator $fB : T \leadsto T(T \otimes T^*)$ given by

$$(fB)(X) = f(a_1, \ldots, a_n)B(X)$$

for any vector field $X$ on an $n$-manifold $M$. 
4. The main result. The main result of this short note is the following classification theorem.

**Theorem 1.** The flow operator $T \otimes T^*$ together with the $k$-th Liouville operators $L^k$ for $k = 0, \ldots, n - 1$ form a basis of the $C^\infty(\mathbb{R}^n)$-module of natural operators $T \rightsquigarrow T(T \otimes T^*)$ over $n$-manifolds.

The proof of Theorem 1 will occupy the rest of this note.

5. The result of J. Dębecki. The vector space $\text{End}(\mathbb{R}^n)$ of all endomorphisms of $\mathbb{R}^n$ is a $\text{GL}(n)$-space because of the usual (adjoint) action of $\text{GL}(n)$ on $\text{End}(\mathbb{R}^n)$.

We have the following result of J. Dębecki.

**Proposition 1 ([1]).** Any $\text{GL}(n)$-equivariant map $C : \text{End}(\mathbb{R}^n) \rightarrow \text{End}(\mathbb{R}^n)$

is of the form

$$C(A) = \sum_{k=0}^{n-1} f_k(a_1(A), \ldots, a_n(A))A^k$$

for some uniquely determined maps $f_k : \mathbb{R}^n \rightarrow \mathbb{R}$.

6. The vertical type natural operators $B : T \rightsquigarrow T(T \otimes T^*)$ over $n$-manifolds. A natural operator $B : T \rightsquigarrow T(T \otimes T^*)$ is of vertical type if $B(X)$ is a vertical vector field for any vector field $X$ on a $n$-manifold.

Using Proposition 1 we prove the following fact.

**Proposition 2.** The $C^\infty(\mathbb{R}^n)$-submodule of vertical type natural operators $B : T \rightsquigarrow T(T \otimes T^*)$ over $n$-manifolds is free and $n$-dimensional. The $k$-th Liouville operators $L^k$ for $k = 0, \ldots, n - 1$ form a basis of this module.

**Proof.** Let $B : T \rightsquigarrow T(T \otimes T^*)$ be a vertical type natural operator over $n$-manifolds. Because of the naturality and the Frobenius theorem this operator is uniquely determined by the restriction of vertical vector field $B(\partial_1)$ to the fiber $\text{End}(T_0 \mathbb{R}^n) = T_0 \mathbb{R}^n \times T_0^* \mathbb{R}^n$.

Using the naturality of $B$ with respect to the homotheties $tid_{\mathbb{R}^n}$ for $t \neq 0$ we see that

$$B(\partial_1)|_{\text{End}(T_0 \mathbb{R}^n)} = B(t\partial_1)|_{\text{End}(T_0 \mathbb{R}^n)}$$

for $t \neq 0$. Putting $t \rightarrow 0$ we see that

$$B(\partial_1)|_{\text{End}(T_0 \mathbb{R}^n)} = B(0)|_{\text{End}(T_0 \mathbb{R}^n)}.$$

Because of the naturality of $B(0)$ with respect to linear automorphisms of $\mathbb{R}^n$ we have a $\text{GL}(n)$-equivariant map

$$C : \text{End}(T_0 \mathbb{R}^n) \rightarrow \text{End}(T_0 \mathbb{R}^n)$$
given by
\[ B(0)(A) = \frac{d}{dt} \bigg|_0 (A + tC(A)) \]
for \( A \in \text{End}(T_0\mathbb{R}^n) \).

By Proposition 1 we have that
\[ C(A) = \sum_{k=0}^{n-1} f_k(a_1(A), \ldots, a_n(A))A^k \]
for some uniquely determined maps \( f_k : \mathbb{R}^n \to \mathbb{R} \). Then
\[ B(\partial_1)(A) = \sum_{k=0}^{n-1} f_k(a_1(A), \ldots, a_n(A))L^k(A) \]
for all \( A \in \text{End}(T_0\mathbb{R}^n) \). That is why \( B = \sum_{k=0}^{n-1} f_kL^k \), as well. \( \square \)

7. Proof of Theorem 1. It is clear that Theorem 1 will be proved after proving the following fact.

**Proposition 3.** Let \( B : T \to T(T \otimes T^*) \) be a natural operator over \( n \)-manifolds. Then there exists a unique map \( f : \mathbb{R}^n \to \mathbb{R} \) such that \( B - fT \otimes T^* \) is a vertical type operator.

Let \( \pi : T\mathbb{R}^n \otimes T^*\mathbb{R}^n \to \mathbb{R}^n \) be the bundle projection.

**Lemma 1.** There exist unique maps \( f_k \in C^\infty(\mathbb{R}^n) \) such that
\[ T\pi(B(w^\circ)(A)) = \sum_{k=0}^{n-1} f_k(a_1(A), \ldots, a_n(A))A^k(w) \]
for \( A \in \text{End}(T_0\mathbb{R}^n) = T_0\mathbb{R}^n \otimes T_0^*\mathbb{R}^n \) and \( w \in T_0\mathbb{R}^n \), where \( w^\circ \) is the "constant" vector field on \( \mathbb{R}^n \) with \( w^\circ_0 = w \).

**Proof.** By the invariance of \( B \) with respect to the homotheties \( tid_{\mathbb{R}^n} \) for \( t \neq 0 \) we have the homogeneity condition
\[ T\pi(B((tw)^\circ)(A)) = tT\pi(B(w^\circ))(A). \]
Then by the homogeneous function theorem, [2], \( T\pi(B(w^\circ))(A) \) depends linearly on \( w \).

So, we can define a map \( C : \text{End}(T_0\mathbb{R}^n) \to \text{End}(T_0\mathbb{R}^n) \) by
\[ C(A)(w) = T\pi(B(w^\circ)(A)) \]
for all \( A \in \text{End}(T_0\mathbb{R}^n) \) and \( w \in T_0\mathbb{R}^n \).

Because of the naturality of \( B \) with respect to linear automorphisms of \( \mathbb{R}^n \), \( C \) is \( GL(n) \)-equivariant. Then applying Proposition 1 we end the proof. \( \square \)

**Lemma 2.** Let \( B : T \to T(T \otimes T^*) \) be as in Lemma 1. Let \( f_0, \ldots, f_{n-1} \) be the maps from Lemma 1. Then \( f_j = 0 \) for \( j = 1, \ldots, n - 1 \).
Proof. Consider $j = 1, \ldots, n-1$. Let $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$. Let $A \in \text{End}(T_0\mathbb{R}^n)$ be such that $A(\partial_i(0)) = \partial_i+1(0)$ for $i = 1, \ldots, n-1$ and $A(\partial_n(0)) = -b_n\partial_1(0) - \ldots - b_1\partial_n(0)$. Then $a_i(A) = b_i$ for $i = 1, \ldots, n$.

Let $\varphi_t = (x^1, \ldots, x^j+1+tx^j+1, \ldots, x^n)$ be the flow of $\partial_j+1+x^j+1\partial_j+1$ near $0 \in \mathbb{R}^n$.

Since $T_0\varphi_1 \circ A \circ T_0\varphi_1^{-1} \neq A$ (as the left hand side evaluated at $\partial_j(0)$ is equal to $2\partial_j+1(0)$ and the right hand side evaluated in the same vector $\partial_j(0)$ is equal to $\partial_j+1(0)$), we have

$$T \otimes T^*(x^j+1\partial_j+1)(A) \neq 0.$$  

Using the Zajtz theorem [4], since $(\partial_j+1+x^j+1\partial_j+1)(0) = \partial_j+1(0) \neq 0$, we find a diffeomorphism $\eta: \mathbb{R} \to \mathbb{R}$ such that

$$j_0^1 \psi = \text{id}$$

and

$$\psi^*\partial_j+1 = \partial_j+1 + x^j+1\partial_j+1$$

near $0 \in \mathbb{R}^n$, where $\psi(x^1, \ldots, x^n) = (x^1, \ldots, x^j, \eta(x^j+1), \ldots, x^n)$.

Clearly $\psi$ preserves $\partial_1$. Because of (2), $\psi$ preserves $A$. Then $\psi$ preserves $B(\partial_j)(A)$.

Because of (2), $\psi$ preserves any vertical vector tangent to $T\mathbb{R}^n \otimes T^*\mathbb{R}^n$ at $A$. Moreover, $\psi$ preserves all $\partial_l$ for $l = 1, \ldots, n$ with $l \neq j+1$. By (3), $\psi$ sends $T \otimes T^*(\partial_j+1)(A)$ into $T \otimes T^*(\partial_j+1+x^j+1\partial_j+1)(A)$. Then $\psi$ sends

$$B(\partial_j)(A) = \sum_{k=0}^{n-1} f_k(a_1(A), \ldots, a_n(A))T \otimes T^*(\partial_k+1)(A) + \text{some vertical vector}$$

into $B(\partial_j)(A) + f_j(b)T \otimes T^*(x^j+1\partial_j+1)(A)$.

Then because of (1), we have $f_j(b) = 0$, as well. \qed

Proof of Proposition 3. Because of Lemmas 1 and 2 we have

$$B(\partial_j)(A) = f_0(a_1(A), \ldots, a_n(A))T \otimes T^*(\partial_j)(A) + \text{some vertical vector}$$

for any $A \in \text{End}(T_0\mathbb{R}^n)$. Since $B$ is determined by $B(\partial_1)$ over $0$, the proof of Proposition 3 is complete. \qed

References


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Received January 4, 2008