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Inclusion properties of certain subclass
of analytic functions defined
by multiplier transformations

Abstract. Let $A$ denote the class of analytic functions with normalization $f(0) = f'(0) - 1 = 0$ in the open unit disk $U = \{z : |z| < 1\}$. Set

$$f_{m,\lambda,\ell}(z) = z + \sum_{k=2}^{\infty} \left( \frac{\ell + 1 + \lambda(k-1)}{\ell + 1} \right)^m z^k \quad (z \in U; m \in \mathbb{N}_0; \lambda \geq 0; \ell \geq 0),$$

and define $f_{m,\lambda,\ell,\mu}$ in terms of the Hadamard product

$$f_{m,\lambda,\ell}(z) \ast f_{m,\lambda,\ell,\mu}(z) = \frac{z}{(1-z)^\mu} \quad (z \in U; \mu > 0).$$

In this paper, we introduce several new subclasses of analytic functions defined by means of the operator $I_{m,\lambda,\ell,\mu}(f)(z) = f_{m,\lambda,\ell,\mu}(z) \ast f(z) \quad (f \in A; m \in \mathbb{N}_0; \lambda \geq 0; \ell \geq 0; \mu > 0)$. Inclusion properties of these classes and the classes involving the generalized Libera integral operator are also considered.

1. Introduction. Let $A$ denote the class of functions of the form:

$$(1.1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. If $f$ and $g$ are analytic in $U$, we say that $f$ is subordinate to $g$, written $f(z) \prec g(z)$, if

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there exists an analytic function \( w \) in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) for \( z \in U \) such that \( f(z) = g(w(z)) \). For \( 0 \leq \eta < 1 \), we denote by \( S^\ast(\eta) \), \( K(\eta) \) and \( C \) the subclasses of \( A \) consisting of all analytic functions which are, respectively, starlike of order \( \eta \), convex of order \( \eta \) and close-to-convex in \( U \) (see, e.g. Srivastava and Owa [18]).

For \( m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), where \( \mathbb{N} = \{1, 2, \ldots\} \), \( \lambda \geq 0 \) and \( \ell \geq 0 \), Cătaş [3] defined the multiplier transformations \( I^m(\lambda, \ell) \) on \( A \) by the following infinite series

\[
I^m(\lambda, \ell) f(z) = z + \sum_{k=2}^{\infty} \frac{(\ell + 1 + \lambda(k - 1))}{\ell + 1} a_k z^k.
\]

It follows from (1.2) that

\[
I^0(\lambda, \ell) = f(z),
\]

\[
(\ell + 1) I^2(\lambda, \ell) f(z) = (\ell + 1 - \lambda) I^1(\lambda, \ell) f(z) + \lambda z (I^1(\lambda, \ell) f(z))',
\]

\[
\lambda > 0, \quad \text{and}
\]

\[
I^{m_1}(\lambda, \ell)(I^{m_2}(\lambda, \ell) f(z)) = I^{m_2}(\lambda, \ell)(I^{m_1}(\lambda, \ell) f(z))
\]

for all integers \( m_1 \) and \( m_2 \).

We note that:

(i) \( I^m(1, \ell) = I^m \) (see Cho and Srivastava [4] and Cho and Kim [5]);

(ii) \( I^m(\lambda, 0) = D^m_\lambda \) (\( m \in \mathbb{N}_0; \lambda \geq 0 \)) (see Al-Oboudi [1]);

(iii) \( I^m(1, 0) = D^m \) (\( m \in \mathbb{N}_0 \)) (see Sălăgean [17]);

(iv) \( I^m(1, 1) = I_m \) (see Uralegaddi and Somanatha [19]).

Let \( S \) be the class of all functions \( \varphi \) which are analytic and univalent in \( U \) and for which \( \varphi(U) \) is convex and \( \varphi(0) = 1 \) and \( \text{Re}\{\varphi(z)\} > 0 \) (\( z \in U \)).

Making use of the principle of subordination between analytic functions, we introduce the subclasses \( S^\ast(\eta; \varphi) \), \( K(\eta; \varphi) \) and \( C(\eta, \delta; \varphi, \psi) \) of the class \( A \) for \( 0 \leq \eta, \delta < 1 \) and \( \varphi, \psi \in S \) (cf., e.g., [6], [8] and [12]), which are defined as follows:

\[
S^\ast(\eta; \varphi) = \left\{ f \in A : \frac{1}{1 - \eta} \left( \frac{zf'(z)}{f(z)} - \eta \right) \prec \varphi(z), \ z \in U \right\},
\]

\[
K(\eta; \varphi) = \left\{ f \in A : \frac{1}{1 - \eta} \left( 1 + \frac{zf''(z)}{f'(z)} - \eta \right) \prec \varphi(z), \ z \in U \right\},
\]

and

\[
C(\eta, \delta; \varphi, \psi) = \left\{ f \in A : \exists g \in S^\ast(\eta, \varphi) \text{ s.t.} \right. \frac{1}{1 - \delta} \left( \frac{zf'(z)}{g(z)} - \delta \right) \prec \psi(z), \ z \in U \right\}.
\]
We note that, for special choices for the functions \( \varphi \) and \( \psi \) in the above definitions we obtain the well-known subclasses of \( A \). For examples, we have

(i) \( S^* \left( \eta; \frac{1+z}{1-z} \right) = S^*(\eta) \quad (0 \leq \eta < 1) \),

(ii) \( K \left( \eta; \frac{1+z}{1-z} \right) = K(\eta) \quad (0 \leq \eta < 1) \)

and

(iii) \( C \left( 0, 0; \frac{1+z}{1-z} : \frac{1+z}{1-z} \right) = C \).

Setting \( f_{\lambda, \ell}^{\mu, z}(z) = z + \sum_{k=2}^{\infty} \left[ \frac{\ell + 1 + \lambda(k-1)}{\ell + 1} \right]^{m} z^{k} \quad (m \in N_0, \lambda \geq 0, \ell \geq 0) \),

we define a new function \( f_{\lambda, \ell}^{\mu, z}(z) \) in terms of the Hadamard product (or convolution) by:

\[
(1.6) \quad f_{\lambda, \ell}^{\mu, z}(z) * f_{\lambda, \ell}^{\mu, z}(z) = \frac{z}{(1-z)^{\mu}} \quad (\mu > 0; z \in U).
\]

Then, motivated essentially by the Choi–Saigo–Srivastava operator [6] (see also [10], [11], [14], and [15]), we now introduce the operators \( f_{\lambda, \ell}^{\mu, z}(z) : A \rightarrow A \), which are defined here by

\[
(1.7) \quad I_{\lambda, \ell}^{\mu, z}(f)(z) = f_{\lambda, \ell}^{\mu, z}(z)
\]

\( (f \in A; m \in N_0; \lambda \geq 0; \ell \geq 0; \mu > 0) \). For a function \( f(z) \in A \), given by

\[
(1.8) \quad I_{\lambda, \ell}^{\mu, z}(f)(z) = z + \sum_{k=2}^{\infty} \left[ \frac{\ell + 1 + \lambda(k-1)}{\ell + 1} \right]^{m} \frac{(\mu)_{k-1}}{(1)_{k-1}} a_k z^k
\]

\( (m \in N_0; \lambda \geq 0; \ell \geq 0; z \in U) \).

We note that:

(i) \( I_{1, 0}^{1, 0}(f)(z) = f(z) \) and \( I_{1}^{0, 2}(f)(z) = z f'(z) \),

and

(ii) \( I_{s, \ell}^{\mu, z}(f)(z) = I_{\ell}^{s, \mu}(f)(z) \) \( (s \in R; \) see Cho and Kim [5]).

In view of (1.8), we obtain the following relations:

\[
(1.9) \quad \lambda z (I_{\lambda, \ell}^{\mu, z}(f)(z))' = (\ell + 1) I_{\lambda, \ell}^{\mu, z}(f)(z) - [\lambda - (\ell + 1)] I_{\lambda, \ell}^{\mu, z}(f)(z)
\]

\( (f \in A; m \in N_0; \lambda > 0; \ell \geq 0; \mu > 0) \) and

\[
(1.10) \quad z (I_{\lambda, \ell}^{\mu, z}(f)(z))' = \mu I_{\lambda, \ell}^{\mu, z+1}(f)(z) - (\mu - 1) I_{\lambda, \ell}^{\mu, z}(f)(z)
\]
Next, by using the operator $I_{\lambda,\ell,\mu}^m$, we introduce the following classes of analytic functions for $\varphi, \psi \in S$, $m \in N_0$, $\lambda \geq 0$, $\ell \geq 0$, $\mu > 0$ and $0 \leq \eta$, $\delta < 1$:

$$S_{\lambda,\ell,\mu}^m(\eta; \varphi) = \{ f \in A : I_{\lambda,\ell,\mu}^m f(z) \in S(\eta; \varphi) \},$$

$$K_{\lambda,\ell,\mu}^m(\eta; \varphi) = \{ f \in A : I_{\lambda,\ell,\mu}^m f(z) \in K(\eta; \varphi) \}$$

and

$$C_{\lambda,\ell,\mu}^m(\eta, \delta; \varphi, \psi) = \{ f \in A : I_{\lambda,\ell,\mu}^m f(z) \in C(\eta, \delta; \varphi, \psi) \}.$$ 

We also have

$$f(z) \in K_{\lambda,\ell,\mu}^m(\eta; \varphi) \iff zf'(z) \in S_{\lambda,\ell,\mu}^m(\eta; \varphi).$$

In particular, we set

$$S_{\lambda,\ell,\mu}^m(\eta; \frac{1 + Az}{1 + Bz})^\alpha = S_{\lambda,\ell,\mu}^m(\eta; A, B, \alpha)$$

and

$$K_{\lambda,\ell,\mu}^m(\eta; \frac{1 + Az}{1 + Bz})^\alpha = K_{\lambda,\ell,\mu}^m(\eta; A, B, \alpha)$$

In this paper, we investigate several inclusion properties of the classes $S_{\lambda,\ell,\mu}^m(\eta; \varphi)$, $K_{\lambda,\ell,\mu}^m(\eta; \varphi)$ and $C_{\lambda,\ell,\mu}^m(\eta, \delta; \varphi, \psi)$ associated with the operator $I_{\lambda,\ell,\mu}^m$. Some applications involving these and other classes of integral operators are also considered.

2. Inclusion properties involving the operator $I_{\lambda,\ell,\mu}^m$. The following lemmas will be required in our investigation.

**Lemma 1** ([7]). Let $\varphi$ be convex, univalent in $U$ with $\varphi(0) = 1$ and $\Re \{ \beta \varphi(z) + \nu \} > 0$ ($\beta, \nu \in C$). If $p$ is analytic in $U$ with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec \varphi(z) \quad (z \in U)$$

implies that

$$p(z) \prec \varphi(z) \quad (z \in U).$$

**Lemma 2** ([13]). Let $\varphi$ be convex, univalent in $U$ and $w$ be analytic in $U$ with $\Re \{ w(z) \} \geq 0$. If $p(z)$ is analytic in $U$ and $p(0) = \varphi(0)$, then

$$p(z) + w(z)p'(z) \prec \varphi(z) \quad (z \in U)$$

implies that

$$p(z) \prec \varphi(z) \quad (z \in U).$$

At first, with the help of Lemma 1, we prove the following theorem.
\textbf{Theorem 1.} Let $m \in N_0$, $\lambda > 0$, $\ell \geq 0$, $\ell + 1 > \lambda$ and $\mu \geq 1$. Then
\[ S_{\lambda, \ell, \mu + 1}^m(\eta; \varphi) \subset S_{\lambda, \ell, \mu}^m(\eta; \varphi) \subset S_{\lambda, \ell, \mu}^{m+1}(\eta; \varphi) \]
$(0 \leq \eta < 1; \phi \in S)$.

\textbf{Proof.} First of all, we will show that
\[ S_{\lambda, \ell, \mu + 1}^m(\eta; \varphi) \subset S_{\lambda, \ell, \mu}^m(\eta; \varphi). \]
Let $f \in S_{\lambda, \ell, \mu + 1}^m(\eta; \varphi)$ and put
\begin{equation}
(2.1) \quad p(z) = \frac{1}{1 - \eta} \left( \frac{z(I_{\lambda, \ell, \mu + 1}^m f(z))'}{I_{\lambda, \ell, \mu + 1}^m f(z)} - \eta \right),
\end{equation}
where $p(z)$ is analytic in $U$ with $p(0) = 1$. Using (1.10) and (2.1), we obtain
\begin{equation}
(2.2) \quad \frac{I_{\lambda, \ell, \mu + 1}^m f(z)}{I_{\lambda, \ell, \mu}^m f(z)} = (1 - \eta)p(z) + \eta + (\mu - 1).
\end{equation}
Differentiating (2.2) logarithmically with respect to $z$, we obtain
\begin{equation}
(2.3) \quad \frac{1}{1 - \eta} \left( \frac{z(I_{\lambda, \ell, \mu + 1}^m f(z))'}{I_{\lambda, \ell, \mu + 1}^m f(z)} - \eta \right) = p(z) + \frac{zp(z)}{(1 - \eta)p(z) + \eta + (\mu - 1)}.
\end{equation}
$(z \in U)$. Applying Lemma 1 to (2.3), it follows that $p \prec \varphi$, that is $f \in S_{\lambda, \ell, \mu}^m(\eta; \varphi)$.

To prove the second part, let $f \in S_{\lambda, \ell, \mu}^m(\eta; \varphi)$ and put
\[ h(z) = \frac{1}{1 - \eta} \left( \frac{z(I_{\lambda, \ell, \mu}^m f(z))'}{I_{\lambda, \ell, \mu}^m f(z)} - \eta \right), \]
where $h$ is analytic in $U$ with $h(0) = 1$. Then, by using the arguments similar to those detailed above with (1.9), it follows that $h \prec \varphi$. This completes the proof of Theorem 1. \hfill \Box

\textbf{Theorem 2.} Let $m \in N_0$, $\lambda > 0$, $\ell \geq 0$, $\ell + 1 > \lambda$ and $\mu \geq 1$. Then
\[ K_{\lambda, \ell, \mu + 1}^m(\eta; \varphi) \subset K_{\lambda, \ell, \mu}^m(\eta; \varphi) \subset K_{\lambda, \ell, \mu}^{m+1}(\eta; \varphi) \]
$(0 \leq \eta < 1; \phi \in S)$.

\textbf{Proof.} Applying (1.11) and Theorem 1, we observe that
\[
\begin{align*}
f & \in K_{\lambda, \ell, \mu + 1}^m(\eta; \varphi) \iff I_{\lambda, \ell, \mu + 1}^m f(z) \in K(\eta; \varphi) \iff z(I_{\lambda, \ell, \mu + 1}^m f(z))' \in S^s(\eta; \varphi) \\
& \iff I_{\lambda, \ell, \mu}^m (zf'(z)) \in S^s(\eta; \varphi) \\
& \iff zf'(z) \in S_{\lambda, \ell, \mu}^m(\eta; \varphi) \\
& \iff zf'(z) \in S_{\lambda, \ell, \mu}^{m+1}(\eta; \varphi) \\
& \iff I_{\lambda, \ell, \mu}^m(zf'(z)) \in S^m(\eta; \varphi)
\end{align*}
\]
\[ \Leftrightarrow z(I_{\lambda,\ell,\mu}^m(zf(z)))' \in S^m(\eta; \varphi) \]
\[ \Leftrightarrow I_{\lambda,\ell,\mu}^m(f(z)) \in K(\eta; \varphi) \]
\[ \Leftrightarrow f(z) \in K_{\lambda,\ell,\mu}^m(\eta; \varphi) \]

and

\[ f(z) \in K_{\lambda,\ell,\mu}^m(\eta; \varphi) \Leftrightarrow zf'(z) \in S^{*}(\eta; \varphi) \]
\[ \Rightarrow zf'(z) \in S^{m+1}_{\lambda,\ell,\mu}(\eta; \varphi) \]
\[ \Leftrightarrow z(I_{\lambda,\ell,\mu}^{m+1}(zf(z)))' \in S^{*}(\eta; \varphi) \]
\[ \Leftrightarrow I_{\lambda,\ell,\mu}^{m+1}(zf(z)) \in K(\eta; \varphi) \]
\[ \Leftrightarrow zf(z) \in K_{\lambda,\ell,\mu}^{m+1}(\eta; \varphi), \]

which evidently proves Theorem 2.

\[ \square \]

Taking \( \varphi(z) = \left( \frac{1 + Az}{1 + Bz} \right)^{\alpha} \) \((-1 \leq B < A \leq 1; 0 < \alpha \leq 1; z \in U)\) in Theorem 1 and Theorem 2, we obtain the following corollary.

**Corollary 1.** Let \( m \in N_0, \lambda > 0, \ell \geq 0, \ell + 1 > \lambda \) and \( \mu \geq 1 \). Then

\[ S_{\lambda,\ell,\mu+1}^m(\eta; A, B; \alpha) \subset S_{\lambda,\ell,\mu}^m(\eta; A, B; \alpha) \subset S_{\lambda,\ell,\mu}^{m+1}(\eta; A, B; \alpha) \]

\( (0 \leq \mu < 1; -1 \leq B < A \leq 1; 0 < \alpha \leq 1), \) and

\[ K_{\lambda,\ell,\mu+1}^m(\eta; A, B; \alpha) \subset K_{\lambda,\ell,\mu}^m(\eta; A, B; \alpha) \subset K_{\lambda,\ell,\mu}^{m+1}(\eta; A, B; \alpha) \]

\( (0 \leq \mu < 1; -1 \leq B < A \leq 1; 0 < \alpha \leq 1). \)

By using Lemma 2, we obtain the following inclusion relation of the class \( C_{\lambda,\ell,\mu}^m(\eta, \delta; \varphi, \psi) \).

**Theorem 3.** Let \( m \in N_0, \lambda > 0, \ell \geq 0, \ell + 1 > \lambda \) and \( \mu \geq 1 \). Then

\[ C_{\lambda,\ell,\mu+1}^m(\eta, \delta; \varphi, \psi) \subset C_{\lambda,\ell,\mu}^m(\eta, \delta; \varphi, \psi) \subset C_{\lambda,\ell,\mu}^{m+1}(\eta, \delta; \varphi, \psi) \]

\( (0 \leq \eta; \delta < 1; \varphi, \psi \in S). \)

**Proof.** We begin by proving that

\[ C_{\lambda,\ell,\mu+1}^m(\eta, \delta; \varphi, \psi) \subset C_{\lambda,\ell,\mu}^m(\eta, \delta; \varphi, \psi). \]

Let \( f \in C_{\lambda,\ell,\mu+1}^m(\eta, \delta; \varphi, \psi) \). Then, in view of the definition of the class \( C_{\lambda,\ell,\mu+1}^m(\eta, \delta; \varphi, \psi) \), there exists a function \( g \in S_{\lambda,\ell,\mu+1}^m(\eta; \varphi) \) such that

\[ \frac{1}{1 - \delta} \left( z(I_{\lambda,\ell,\mu+1}^m(\eta; \varphi) - \delta) \right) \prec \psi(z) \quad (z \in U). \]
Now let
\[
p(z) = \frac{1}{1 - \delta} \left( \frac{z(I_{\lambda,\ell,\mu}^m f(z))'}{I_{\lambda,\ell,\mu}^m g(z)} - \delta \right),
\]
where \( p \) is analytic in \( U \) with \( p(0) = 1 \). Using the identity (1.10), we obtain
\[
(2.4) \quad [(1 - \delta)p(z) + \delta]I_{\lambda,\ell,\mu}^m g(z) + (\mu - 1)I_{\lambda,\ell,\mu}^m f(z) = \mu I_{\lambda,\ell,\mu+1}^m f(z).
\]
Differentiating (2.4) with respect to \( z \) and multiplying by \( z \), we have
\[
(2.5) \quad (1 - \delta)zp'(z)I_{\lambda,\ell,\mu}^m (z) + [(1 - \delta)p(z) + \delta]z(I_{\lambda,\ell,\mu}^m g(z))' = \mu z((I_{\lambda,\ell,\mu+1}^m f(z))' - (\mu - 1)z(I_{\lambda,\ell,\mu}^m f(z))').
\]
Since \( g \in S_{\lambda,\ell,\mu+1}^m (\eta; \varphi) \), by Theorem 1, we know that \( g \in S_{\lambda,\ell,\mu}^m (\eta; \varphi) \). Let
\[
q(z) = \frac{1}{1 - \eta} \left( \frac{z(I_{\lambda,\ell,\mu}^m g(z))'}{I_{\lambda,\ell,\mu}^m g(z)} - \eta \right).
\]
Then, using the identity (1.10) once again, we obtain
\[
(2.6) \quad \mu \frac{I_{\lambda,\ell,\mu+1}^m f(z)}{I_{\lambda,\ell,\mu}^m g(z)} = (1 - \eta)q(z) + \eta + (\mu - 1).
\]
From (2.5) and (2.6), we have
\[
\frac{1}{1 - \delta} \left( \frac{z(I_{\lambda,\ell,\mu+1}^m f(z))'}{I_{\lambda,\ell,\mu+1}^m g(z)} - \delta \right) = p(z) + \frac{zp'(z)}{(1 - \eta)q(z) + \eta + (\mu - 1)}.
\]
Since \( 0 \leq \eta < 1 \), \( \mu \geq 1 \) and \( q < \varphi \) in \( U \),
\[
\text{Re} \left\{ (1 - \eta)q(z) + \eta + \mu - 1 \right\} > 0
\]
\( (z \in U) \). Hence applying Lemma 2, we can show that \( p \prec \psi \), so that \( f \in C_{\lambda,\ell,\mu}^m (\eta; \delta; \varphi; \psi) \).

For the second part, by using the arguments similar to those detailed above with (1.9), we obtain
\[
C_{\lambda,\ell,\mu}^m (\eta; \delta; \varphi; \psi) \subset C_{\lambda,\ell,\mu}^{m+1} (\eta; \delta; \varphi; \psi).
\]
This completes the proof of Theorem 3. \( \square \)

3. Inclusion properties involving the integral operator \( F_c \). In this section, we consider the generalized Libera integral operator \( F_c \) (see [16], [2] and [9]) defined by
\[
(3.1) \quad F_c(f) = F_c(f)(z) = \frac{c + 1}{z} \int_0^z \frac{t^{c-1}f(t)}{a^c} dt
\]
\( (c > -1; \ f \in A) \). We first prove the following theorem.

**Theorem 4.** Let \( c, \lambda \geq 0, \ m \in \mathbb{N}_0, \ \ell \geq 0 \) and \( \mu > 0 \). If \( f \in S_{\lambda,\ell,\mu}^m (\eta; \varphi) (0 \leq \eta < 1; \ \varphi \in S) \), then \( F_c(f) \in S_{\lambda,\ell,\mu}^m (\eta; \varphi) (0 \leq \eta < 1; \ \varphi \in S) \).
Proof. Let $f \in S_{\lambda, \ell, \mu}^m(\eta; \varphi)$ and put

\begin{equation}
 p(z) = \frac{1}{1 - \eta} \left( \frac{z(I_{\lambda, \ell, \mu}^m f_c(z))'}{I_{\lambda, \ell, \mu}^m f_c(z)} - \eta \right),
\end{equation}

where $p$ is analytic in $U$ with $p(0) = 1$. From (3.1), we have

\begin{equation}
 z(I_{\lambda, \ell, \mu}^m f_c(z))' = (c + 1)I_{\lambda, \ell, \mu}^m f(z) - cI_{\lambda, \ell, \mu}^m f_c(f(z)).
\end{equation}

Then, by using (3.2) and (3.3), we have

\begin{equation}
 (c + 1) \frac{I_{\lambda, \ell, \mu}^m f(z)}{I_{\lambda, \ell, \mu}^m f_c(f(z))} = (1 - \eta)p(z) + \eta + c.
\end{equation}

Differentiating (3.4) logarithmically with respect to $z$ and multiplying by $z$, we have

\begin{equation}
 p(z) + \frac{zp'(z)}{(1 - \eta)p(z) + \eta + c} = \frac{1}{1 - \eta} \left( \frac{z(I_{\lambda, \ell, \mu}^m f(z))'}{I_{\lambda, \ell, \mu}^m f(z)} - \eta \right) \quad (z \in U).
\end{equation}

Hence, by virtue of Lemma 1, we conclude that $p \prec \varphi$ ($z \in U$), which implies that $F_c(f) \in S_{\lambda, \ell, \mu}^m(\eta; \varphi)$. \(\square\)

Next, we derive an inclusion property involving $F_c$, which is given by the following theorem.

**Theorem 5.** Let $c, \ell \geq 0$, $m \in N_0$, $\lambda \geq 0$ and $\mu > 0$. If $f \in K_{\lambda, \ell, \mu}^m(\eta; \varphi)$ ($0 \leq \eta < 1; \varphi \in S$), then $F_c(f) \in K_{\lambda, \ell, \mu}^m(\eta; \varphi)$ ($0 \leq \eta < 1; \varphi \in S$).

**Proof.** By applying Theorem 4, it follows that

\begin{align*}
 f(z) \in K_{\lambda, \ell, \mu}^m(\eta; \varphi) \iff z f'(z) \in S_{\lambda, \ell, \mu}^m(\eta; \varphi) \\
 \Rightarrow F_c(z f'(z)) \in S_{\lambda, \ell, \mu}^m(\eta; \varphi) \\
 \iff z(F_c(f(z))' \in S_{\lambda, \ell, \mu}^m(\eta; \varphi) \\
 \iff F_c(f(z)) \in K_{\lambda, \ell, \mu}^m(\eta; \varphi),
\end{align*}

which proves Theorem 5. \(\square\)

From Theorem 4 and Theorem 5, we have the following corollary.

**Corollary 2.** Let $c, \ell \geq 0$, $m \in N_0$, $\lambda > 0$ and $\mu > 0$. If $f \in S_{\lambda, \ell, \mu}^m(\eta; A, B; \alpha)$ (or $K_{\lambda, \ell, \mu}^m(\eta; A, B; \alpha)$) ($0 \leq \eta < 1; -1 \leq B < A \leq 1; 0 < \alpha \leq 1$), then $F_c(f)$ belongs to the class $S_{\lambda, \ell, \mu}^m(\eta; A, B; \alpha)$ (or $K_{\lambda, \ell, \mu}^m(\eta; A, B; \alpha)$) ($0 \leq \eta < 1; -1 \leq B < A \leq 1; 0 < \alpha \leq 1$).

Finally, we prove the following theorem.

**Theorem 6.** Let $c, \ell \geq 0$, $m \in N_0$, $\lambda > 0$ and $\mu > 0$. If $f \in C_{\lambda, \ell, \mu}^m(\eta; \delta; \varphi; \psi)$ ($0 \leq \eta; \delta < 1; \varphi, \psi \in S$), then $F_c(f) \in C_{\lambda, \ell, \mu}^m(\eta; \delta; \varphi; \psi)$ ($0 \leq \eta; \delta < 1; \varphi, \psi \in S$).
Proof. Let \( f \in C^m_{\lambda,\ell,\mu}(\eta; \delta; \varphi; \psi) \). Then, in view of the definition of the class \( C^m_{\lambda,\ell,\mu}(\eta; \delta; \varphi; \psi) \), there exists a function \( g \in S^m_{\lambda,\ell,\mu}(\eta; \varphi) \) such that

\[
\frac{1}{1 - \delta} \left( \frac{z(I^m_{\lambda,\ell,\mu}f(z))'}{I^m_{\lambda,\ell,\mu}g(z)} - \delta \right) \prec \psi(z) \quad (z \in U).
\]

Thus, we put

\[
p(z) = \frac{1}{1 - \delta} \left( \frac{z(I^m_{\lambda,\ell,\mu}F_c(f)(z))'}{I^m_{\lambda,\ell,\mu}F_c(g)(z)} - \delta \right),
\]

where \( p \) is analytic in \( U \) with \( p(0) = 1 \). Since \( g \in S^m_{\lambda,\ell,\mu}(\eta; \varphi) \), we see from Theorem 4 that \( F_c(g) \in S^m_{\lambda,\ell,\mu}(\eta; \varphi) \). Using (3.3), we have

\[
[(1 - \delta)p(z) + \delta] I^m_{\lambda,\ell,\mu}F_c(g)(z) + cI^m_{\lambda,\ell,\mu}F_c(f)(z) = (c + 1)I^m_{\lambda,\ell,\mu}f(z).
\]

Then, by a simple calculations, we get

\[
(c + 1) \frac{z(I^m_{\lambda,\ell,\mu}f(z))'}{I^m_{\lambda,\ell,\mu}F_c(g)(z)} = [(1 - \delta)p(z) + \delta] [(1 - \eta)q(z) + \eta + c] + (1 - \delta)zp'(z),
\]

where

\[
q(z) = \frac{1}{1 - \eta} \left( \frac{z(I^m_{\lambda,\ell,\mu}F_c(g)(z))'}{I^m_{\lambda,\ell,\mu}F_c(g)(z)} - \eta \right).
\]

Hence, we have

\[
\frac{1}{1 - \delta} \left( \frac{z(I^m_{\lambda,\ell,\mu}f(z))'}{I^m_{\lambda,\ell,\mu}g(z)} - \delta \right) = p(z) + \frac{zp'(z)}{(1 - \eta)q(z) + \eta + c}.
\]

The remaining part of the proof of Theorem 6 is similar to that of Theorem 3 and so we omit it.

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