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On limiting values of Cauchy type integral in a harmonic algebra with two-dimensional radical

Abstract. We consider a certain analog of Cauchy type integral taking values in a three-dimensional harmonic algebra with two-dimensional radical. We establish sufficient conditions for an existence of limiting values of this integral on the curve of integration.

1. Introduction. Let \( \Gamma \) be a closed Jordan rectifiable curve in the complex plane \( \mathbb{C} \). By \( D^+ \) and \( D^- \) we denote, respectively, the interior and the exterior domains bounded by the curve \( \Gamma \).

N. Davydov [1] established sufficient conditions for an existence of limiting values of the Cauchy type integral

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{g(t)}{t - \xi} dt, \quad \xi \in \mathbb{C} \setminus \Gamma,
\]

on \( \Gamma \) from the domains \( D^+ \) and \( D^- \). This result stimulated a development of the theory of Cauchy type integral on curves which are not piecewise-smooth.

In particular, using the mentioned result of the paper [1], the following result was proved: if the curve \( \Gamma \) satisfies the condition (see [2])

\[
\theta(\varepsilon) := \sup_{\xi \in \Gamma} \theta_{\varepsilon}(\xi) = O(\varepsilon), \quad \varepsilon \to 0
\]

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(here $\theta_\epsilon(\xi) := \text{mes}\{t \in \Gamma : |t - \xi| \leq \epsilon\}$, where mes denotes the linear Lebesgue measure on $\Gamma$), and the modulus of continuity

$$
\omega_g(\epsilon) := \sup_{t_1, t_2 \in \Gamma, |t_1 - t_2| \leq \epsilon} |g(t_1) - g(t_2)|
$$

of a function $g : \Gamma \to \mathbb{C}$ satisfies the Dini condition

$$
(3) \quad \int_0^1 \frac{\omega_g(\eta)}{\eta} d\eta < \infty,
$$

then the integral (1) has limiting values in every point of $\Gamma$ from the domains $D^+$ and $D^-$ (see [3]). The condition (2) means that the measure of a part of the curve $\Gamma$ in every disk centered at a point of the curve is commensurable with the radius of the disk.

In this paper we consider a certain analogue of Cauchy type integral taking values in a three-dimensional harmonic algebra with two-dimensional radical and study the question about an existence of its limiting values on the curve of integration.

2. A three-dimensional harmonic algebra with a two-dimensional radical. Let $A_3$ be a three-dimensional commutative associative Banach algebra with unit $1$ over the field of complex numbers $\mathbb{C}$. Let $\{1, \rho_1, \rho_2\}$ be a basis of algebra $A_3$ with the multiplication table:

$\rho_1 \rho_2 = \rho_2 \rho_1 = 0, \quad \rho_2 \rho_2 = \rho_2.$

$A_3$ is a harmonic algebra, i.e. there exists a harmonic basis $\{e_1, e_2, e_3\} \subset A_3$ satisfying the following conditions (see [5], [6], [7], [8], [9]):

$$
(4) \quad e_1^2 + e_2^2 + e_3^2 = 0, \quad e_j^2 \neq 0 \text{ for } j = 1, 2, 3.
$$

P. Ketchum [5] discovered that every function $\Phi(\zeta)$ analytic with respect to the variable $\zeta := xe_1 + ye_2 + ze_3$ with real $x, y, z$ satisfies the equalities

$$
(5) \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \Phi(\zeta) = \Phi''(\zeta) (e_1^2 + e_2^2 + e_3^2) = 0
$$

owing to the equality (4). I. Mel’nichenko [7] noticed that doubly differentiable in the sense of Gateaux functions form the largest class of functions $\Phi$ satisfying the equalities (5).

All harmonic bases in $A_3$ are constructed by I. Mel’nichenko in [9].

Consider a harmonic basis

$$
e_1 = 1, \quad e_2 = i + \frac{1}{2} i \rho_2, \quad e_3 = -\rho_1 - \frac{\sqrt{3}}{2} i \rho_2
$$
in $A_3$ and the linear envelope $E_3 := \{\zeta = xe_1 + ye_2 + ze_3 : x, y, z \in \mathbb{R}\}$ over the field of real numbers $\mathbb{R}$, that is generated by the vectors $1, e_2, e_3$. Associate with a domain $\Omega \subset \mathbb{R}^3$ the domain $\Omega_\zeta := \{\zeta = xe_1 + ye_2 + ze_3 : (x, y, z) \in \Omega\}$ in $E_3$. 
The algebra $A_3$ have the unique maximal ideal $\{\lambda_1 \rho_1 + \lambda_2 \rho_2 : \lambda_1, \lambda_2 \in \mathbb{C}\}$ which is also the radical of $A_3$. Thus, it is obvious that the straight line $\{2e_3 : z \in \mathbb{R}\}$ is contained in the radical of algebra $A_3$.

$A_3$ is a Banach algebra with the Euclidean norm

$$\|a\| := \sqrt{|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2},$$

where $a = \xi_1 + \xi_2 e_2 + \xi_3 e_3$ and $\xi_1, \xi_2, \xi_3 \in \mathbb{C}$.

We say that a continuous function $\Phi : \Omega \rightarrow A_3$ is monogenic in a domain $\Omega \subset E_3$ if $\Phi$ is differentiable in the sense of Gateaux in every point of $\Omega$, i.e. if for every $\zeta \in \Omega$ there exists $\Phi'(\zeta) \in A_3$ such that

$$\lim_{\varepsilon \rightarrow 0^+} (\Phi(\zeta + \varepsilon h) - \Phi(\zeta)) \varepsilon^{-1} = h\Phi'(\zeta) \quad \forall h \in E_3.$$

For monogenic functions $\Phi : \Omega \rightarrow A_3$ we established basic properties analogous to properties of analytic functions of the complex variable: the Cauchy integral theorem, the Cauchy integral formula, the Morera theorem, the Taylor expansion (see [11]).

3. On existence of limiting values of a hypercomplex analogue of the Cauchy type integral. In what follows, $t_1, t_2, x, y, z \in \mathbb{R}$ and the variables $x, y, z$ with subscripts are real. For example, $x_0$ and $x_1$ are real, etc.

Let $\Gamma_\zeta := \{\tau = t_1 + t_2 e_2 : t_1 + i t_2 \in \Gamma\}$ be the curve congruent to the curve $\Gamma \subset \mathbb{C}$. Consider the domain $\Pi_\zeta^\pm := \{\zeta = x + ye_2 + ze_3 : x + iy \in D^\pm, z \in \mathbb{R}\}$ in $E_3$. By $\Sigma_\zeta$ we denote the common boundary of domains $\Pi_\zeta^+$ and $\Pi_\zeta^-$. Consider the integral

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_{\Gamma_\zeta} \varphi(\tau)(\tau - \zeta)^{-1} d\tau$$

with a continuous density $\varphi : \Gamma_\zeta \rightarrow \mathbb{R}$. The function (6) is monogenic in the domains $\Pi_\zeta^+$ and $\Pi_\zeta^-$, but the integral (6) is not defined for $\zeta \in \Sigma_\zeta$.

For the function $\varphi : \Gamma_\zeta \rightarrow \mathbb{R}$ consider the modulus of continuity

$$\omega_\varphi(\varepsilon) := \sup_{\tau_1, \tau_2 \in \Gamma_\zeta, \|\tau_1 - \tau_2\| \leq \varepsilon} |\varphi(\tau_1) - \varphi(\tau_2)|,$$

and a singular integral

$$\int_{\Gamma_\zeta} \left(\varphi(\tau) - \varphi(\zeta_0)\right)(\tau - \zeta_0)^{-1} d\tau := \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\zeta \setminus \Gamma_\zeta^\varepsilon(\zeta_0)} \left(\varphi(\tau) - \varphi(\zeta_0)\right)(\tau - \zeta_0)^{-1} d\tau,$$

where $\zeta_0 \in \Gamma_\zeta$ and $\Gamma_\zeta^\varepsilon(\zeta_0) := \{\tau \in \Gamma_\zeta : \|\tau - \zeta_0\| \leq \varepsilon\}$.

Below, in Theorem 1 in the case where the curve $\Gamma$ satisfies the condition (2) and the modulus of continuity of the function $\varphi$ satisfies a condition of the type (3), we establish the existence of certain limiting values of the integral (6) in points $\zeta_0 \in \Gamma_\zeta$ when $\zeta$ tends to $\zeta_0$ from $\Pi_\zeta^+$ or $\Pi_\zeta^-$ along
a curve that is not tangential to the surface $\Sigma$ outside of the plane of
curve $\Gamma$.

For the Euclidean norm in $\mathbb{R}^3$ the following inequalities are fulfilled:

$$(7) \quad \|ab\| \leq 2\sqrt{14}\|a\||\|b\| \quad \forall a, b \in \mathbb{R}^3,$$

$$(8) \quad \left\| \int_{\Gamma} \psi(\tau)d\tau \right\| \leq 9M \int_{\Gamma} \|\psi(\tau)\|d\tau$$

with the constant $M := \max\{1, \|e_2\|, \|e_2e_3\|, \|e_3^2\|\}$ for any measurable set
$\Gamma' \subset \Gamma$ and all continuous functions $\psi : \Gamma' \to \mathbb{R}^3$.

**Lemma 1.** Let $\Gamma$ be a closed Jordan rectifiable curve satisfying the condition
(2) and the modulus of continuity of a function $\varphi : \Gamma \to \mathbb{R}$ satisfies the
condition of the type (3). If a point $\zeta$ tends to $\zeta_0 \in \Gamma$ along a curve $\gamma$ for
which there exists a constant $m < 1$ such that the inequality

$$(9) \quad |z| \leq m\|\zeta - \zeta_0\|$$

is fulfilled for all $\zeta = x + ye_2 + z e_3 \in \gamma$, then

$$\lim_{\zeta \to \zeta_0, \zeta \in \gamma} \int_{\Gamma} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta)^{-1}d\tau = \int_{\Gamma} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta_0)^{-1}d\tau.$$ 

**Proof.** Let $\varepsilon := \|\zeta - \zeta_0\|$. Consider the difference

$$\int_{\Gamma} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta)^{-1}d\tau - \int_{\Gamma} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta_0)^{-1}d\tau$$

$$= \int_{\Gamma} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta)^{-1}d\tau - \int_{\Gamma} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta_0)^{-1}d\tau$$

$$+ (\zeta - \zeta_0) \int_{\Gamma} \left( \varphi(\tau) - \varphi(\zeta_0) \right)(\tau - \zeta)^{-1}(\tau - \zeta_0)^{-1}d\tau =: I_1 - I_2 + I_3.$$

To estimate $I_1$ we choose a point $\zeta_1 = x_1 + y_1 e_2$ on $\Gamma$ such that $\|\zeta - \zeta_1\| = \min_{\tau \in \Gamma} \|\tau - \zeta\|$. Using the inequalities (7) and (8), we obtain

$$\|I_1\| = \left\| \int_{\Gamma} \left( \varphi(\tau) - \varphi(\zeta_1) \right)(\tau - \zeta)^{-1}d\tau + \left( \varphi(\zeta_1) - \varphi(\zeta_0) \right)(\tau - \zeta_1)^{-1}d\tau \right\|$$

$$\leq 18\sqrt{14}M \int_{\Gamma} |\varphi(\tau) - \varphi(\zeta_1)| \|\tau - \zeta\|^{-1} \|d\tau\|$$
+|\varphi(\zeta_1) - \varphi(\zeta_0)| \left\| \int_{\Gamma_\zeta(\zeta_0)} (\tau - \zeta)^{-1} d\tau \right\| =: I_1' + I_1''.

It follows from Lemma 1.1 [9] that

\begin{align*}
(10) \quad (\tau - \zeta)^{-1} &= \frac{1}{t - \xi} - \frac{z}{(t - \xi)^2} \rho_1 + \left( \frac{i}{2} \frac{y - t_2 - \sqrt{3}z}{(t - \xi)^2} + \frac{z^2}{(t - \xi)^3} \right) \rho_2
\end{align*}

for all \( \zeta = x + ye^2 + ze^3 \in \Pi_\zeta \) and \( \tau = t_1 + t_2 e^2 \in \Gamma_\zeta \), where \( \xi := x + iy \) and \( t := t_1 + it_2 \). The following inequality follows from the relations (9) and (10):

\begin{align*}
(11) \quad \| (\tau - \zeta)^{-1} \| \leq c(m) \frac{1}{|t - \xi|},
\end{align*}

where the constant \( c(m) \) depends only on \( m \).

Using the inequality \( |t - \xi| \geq |t - \xi_1|/2 \) with \( \xi_1 := x_1 + iy_1 \) and the inequality (11), we obtain:

\begin{align*}
\| I_1' \| &\leq 18\sqrt{14} M c(m) \int_{\Gamma_\zeta(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_1)|}{|t - \xi|} |d\tau| \\
&\leq 36\sqrt{14} M c(m) \int_{\Gamma_\zeta(\zeta_0)} \frac{|\varphi(\tau) - \varphi(\zeta_1)|}{|t - \xi_1|} |d\tau| \\
&\leq 36\sqrt{14} M c(m) \int_{[0,4\varepsilon]} \frac{\omega(\eta)}{\eta} d\theta_{\xi_1}(\eta),
\end{align*}

where the last integral is understood as a Lebesgue–Stieltjes integral.

To estimate the last integral we use Proposition 1 [10] (see also the proof of Theorem 1 [4]) and the condition (2). So, we have

\begin{align*}
\int_{[0,4\varepsilon]} \frac{\omega(\eta)}{\eta} d\theta_{\xi_1}(\eta) &\leq \int_{0}^{8\varepsilon} \frac{\theta_{\xi_1}(\eta)}{\eta^2} \omega(\eta) d\eta \leq c \int_{0}^{8\varepsilon} \frac{\omega(\eta)}{\eta^2} d\eta \to 0, \quad \varepsilon \to 0,
\end{align*}

where the constant \( c \) does not depend on \( \varepsilon \).

To estimate \( I_1'' \) we introduce the domain \( D_{\zeta}(\zeta_0) := \{ \tau = t_1 + t_2 e^2 : t_1 + it_2 \in D^+, \| \tau - \zeta_0 \| \leq 2\varepsilon \} \) and its boundary \( \partial D_{\zeta}(\zeta_0) \). Using the inequalities (8) and (11), we obtain:
\[ \| I_1'' \| \leq \omega_\varphi (\| \zeta_1 - \zeta_0 \|) \left\| \int_{\Gamma^{2\epsilon}_\zeta (\zeta_0)} (\tau - \zeta)^{-1} d\tau \right\| \]

\[ = \omega_\varphi (\| \zeta_1 - \zeta_0 \|) \left[ \int_{\partial D^{2\epsilon}_\zeta (\zeta_0)} (\tau - \zeta)^{-1} d\tau - \int_{\partial D^{2\epsilon}_\zeta (\zeta_0) \setminus \Gamma^{2\epsilon}_\zeta (\zeta_0)} (\tau - \zeta)^{-1} d\tau \right] \]

\[ \leq \omega_\varphi (\| \zeta_1 - \zeta_0 \|) \left( 2\pi + 9Mc(m) \int_{\partial D^{2\epsilon}_\zeta (\zeta_0) \setminus \Gamma^{2\epsilon}_\zeta (\zeta_0)} \| d\tau \| / |t - \zeta| \right) \]

\[ \leq \omega_\varphi (2\epsilon) \left( 2\pi + 9Mc(m) \frac{1}{\varepsilon} \frac{1}{2\pi 2\epsilon} \right) \rightarrow 0, \quad \varepsilon \rightarrow 0. \]

Estimating \( I_2 \), by analogy with the estimation of \( I_1' \), we obtain:

\[ \| I_2 \| \leq c \int \frac{\omega_\varphi (\eta)}{\eta} d\eta \rightarrow 0, \quad \varepsilon \rightarrow 0, \]

where the constant \( c \) does not depend on \( \varepsilon \).

Using the inequality \(|t - \xi| \geq |t - \xi_0|/2\), where the point \( \xi_0 := x_0 + iy_0 \) corresponds to the point \( \zeta_0 = x_0 + y_0e_2 \), and using the relations (7), (8), (11) and (2), by analogy with the estimation of \( I_1' \), we obtain:

\[ \| I_3 \| \leq 9M(2\sqrt{4})^2 \varepsilon \int_{\Gamma^{\epsilon}(\zeta_0)} |\varphi(\tau) - \varphi(\zeta_0)| \left\| (\tau - \zeta)^{-1} \right\| \left\| (\tau - \zeta_0)^{-1} \right\| d\tau \]

\[ \leq c \varepsilon \int_{\Gamma^{\epsilon}(\zeta_0)} \left| \frac{\varphi(\tau) - \varphi(\zeta_0)}{|t - \zeta_0|} \right| \left\| (\tau - \zeta)^{-1} \right\| d\tau \]

\[ \leq c \varepsilon \int_{\Gamma^{\epsilon}(\zeta_0)} \omega_\varphi (\eta) \frac{d\theta_{\xi_0}(\eta)}{\eta^2} \leq c \varepsilon \int_{\Gamma^{\epsilon}(\zeta_0)} \frac{2d|\xi_0(\eta)|\omega_\varphi (\eta)}{\eta^3} d\eta \]

\[ \leq c \varepsilon \int_{2\epsilon} \frac{2d\omega_\varphi (\eta)}{\eta^2} d\eta \rightarrow 0, \quad \varepsilon \rightarrow 0, \]

where \( d := \max_{\xi_1, \xi_2 \in \Gamma} |\xi_1 - \xi_2| \) is the diameter of \( \Gamma \) and \( c \) denotes different constants which do not depend on \( \varepsilon \). The lemma is proved. 

Let \( \hat{\Phi}^{\pm}(\zeta_0) \) be the boundary value of function (6) when \( \zeta \) tends to \( \zeta_0 \in \Gamma_\zeta \) along a curve \( \gamma_\zeta \) for which there exists a constant \( m < 1 \) such that the inequality (9) is fulfilled for all \( \zeta = x + ye_2 + z\zeta_3 \in \gamma_\zeta \).
Theorem 1. Let $\Gamma$ be a closed Jordan rectifiable curve satisfying the condition (2) and the modulus of continuity of a function $\varphi : \Gamma \to \mathbb{R}$ satisfies the condition of the type (3). Then the integral (6) has boundary values $\hat{\Phi}^\pm(\zeta_0)$ for all $\zeta_0 \in \Gamma$ that are expressed by the formulas:

$$\hat{\Phi}^+(\zeta_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\varphi(\tau) - \varphi(\zeta_0))(\tau - \zeta_0)^{-1}d\tau + \varphi(\zeta_0)}{\Gamma}$$

$$\hat{\Phi}^-(\zeta_0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{(\varphi(\tau) - \varphi(\zeta_0))(\tau - \zeta_0)^{-1}d\tau}{\Gamma}.$$ 

The theorem follows from the Lemma 1 and the equalities

$$\frac{1}{2\pi i} \int_{\Gamma} \varphi(\tau)(\tau - \zeta)^{-1}d\tau = \frac{1}{2\pi i} \int_{\Gamma} (\varphi(\tau) - \varphi(\zeta_0))(\tau - \zeta)^{-1}d\tau + \varphi(\zeta_0) \quad \forall \zeta \in \Pi^+_\zeta,$$

$$\frac{1}{2\pi i} \int_{\Gamma} \varphi(\tau)(\tau - \zeta)^{-1}d\tau = \frac{1}{2\pi i} \int_{\Gamma} (\varphi(\tau) - \varphi(\zeta_0))(\tau - \zeta)^{-1}d\tau \quad \forall \zeta \in \Pi^-_\zeta.$$ 

In comparison with Theorem 1, note that additional assumptions about the function $\varphi$ are required for an existence of limiting values of the function (6) from $\Pi^+_\zeta$ or $\Pi^-_\zeta$ on the boundary $\Sigma_\zeta$. We are going to state these results in next papers.

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